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Generalizing the Baer–Kaplansky Theorem

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Abstract

By reformulating the Baer–Kaplansky Theorem it is shown that it holds for large classes of modules over arbitrary rings. New classes of modules satisfying the classical Baer–Kaplansky Theorem are found.

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The Baer–Kaplansky Theorem states that two abelian torsion groups are isomorphic if, and only if, their endomorphism rings are isomorphic. Moreover, in that case the isomorphism between their endomorphism rings is induced, in the obvious way, by an isomorphism between the groups. This result has inspired much interest and much research effort has been expended to extend it to more general situations including that of modules over arbitrary rings. (We refer the reader to [8] for an extensive survey which places the Baer–Kaplansky result in the context of a broader and older area of research.)

For modules over noncommutative rings it fails trivially. For example, let the base ring be the ring of upper triangular 2×2-matrices over a field. This ring has two non-isomorphic simple left modules and an indecomposable projective left ideal of length two. But the endomorphism ring of each of these is simply the field.

It seems that to extend the theorem to noncommutative rings the hypotheses would need some modification. Since an abelian torsion group is the direct sum of its p-components, p a prime, and there are no homomorphisms between different components, Baer [2] and Kaplansky [7] proved their results for p-groups. An indecomposable p-group is either the cyclic group $\mathbb{Z}(p^n)$ or the quasicyclic group $\mathbb{Z}(p^\infty)$. The endomorphism ring of the former is itself (as a ring) and of the latter it is the ring of p-adic integers. So, clearly, isomorphisms between endomorphism rings of abelian torsion groups "preserve" indecomposables. (A precise definition of this concept, which we call *IP-isomorphism*, is given before Proposition 1.) It is therefore, a priori, not

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unreasonable to impose this condition on the isomorphisms between the rings. It is the aim of this note to investigate what the condition entails.

Baer proved his result for groups which are direct sums of p-cycles of bounded orders (with a slight restriction). Kaplansky proved the general case by proving it for primary modules over (commutative) complete discrete valuation rings. We study modules which are direct sums of indecomposables. We show that there is a generalized Baer–Kaplansky Theorem for all such modules with some additional conditions. We present evidence which strongly suggests that if a ring has non-central idempotents then no reasonably diverse class of its modules will satisfy the (original) Baer–Kaplansky Theorem.

A by-product of our approach is that new light is shed on the existing theory and interesting new questions arise. In particular, it is shown that most, but not all, of the required information is carried by the endomorphism rings. New classes of rings satisfying the (original) Baer–Kaplansky Theorem are found. It appears that these are the first in which the idempotents of the ring do not act in unison on the modules.

Throughout this note, R will be a ring with unity, all ideals and modules will be nonzero unital left R-modules. The letter M will denote an R-module and e will always be an idempotent.

Let $M = \bigoplus_{\Delta} I_\delta$ be a decomposition into indecomposables. Let N be a submodule of M. We say N is *finitely embedded* in (the above decomposition of) M if it is contained in the sum of a finite number of the I_δ. We say that M has the *finite embedding property* (with respect to the above decomposition) if every indecomposable summand of M is finitely embedded (in the above decomposition). It is easy to verify that all finitely generated submodules are finitely embedded.

A ring isomorphism $\Phi : End(M) \to End(N)$ is an *IP-isomorphism* if for every primitive idempotent $e \in End(M)$, $N(e\Phi) \cong Me$.

Proposition 1. *Let M have the finite embedding property with respect to a decomposition into indecomposables and let N be generated by indecomposable summands. Then $M \cong N$ if, and only if, there is an IP-isomorphism $End(M) \to End(N)$.*

Proof. Let $M = \bigoplus_{\Delta} I_\delta$ be a decomposition into indecomposables with respect to which M has the finite embedding property. Let $\Phi : End(M) \to End(N)$ be an IP-isomorphism, $e_\delta \in End(M)$ the projection $M \to I_\delta$ and let $f_\delta \in End(N)$ be the image of e_δ under Φ. Since Φ preserves indecomposables, $Me_\delta \cong Nf_\delta$. Let $\Psi : M \to N$ be a homomorphism which maps each $I_\delta = Me_\delta$ isomorphically onto Nf_δ.

Since the e_δ are mutually orthogonal then so are the f_δ and therefore the Nf_δ generate a direct sum. Hence, Ψ is monic with image $\bigoplus_{\Delta} Nf_\delta$.

To see that Ψ is epic, let J be any indecomposable summand of N and f the idempotent projection $N \to J$. Let e be the pre-image of f under Φ. Since f is primitive, e is primitive and so Me is an indecomposable summand of M and is contained in a finite sum $I_1 \oplus \cdots \oplus I_n$ of the I_δ. Then $1 - e_1 - \cdots - e_n$ is a right annihilator for e and consequently $1 - f_1 - \cdots - f_n$ is a right annihilator for f. Therefore, $f_1 + \cdots + f_n$

is a right identity for J which means that $J \subseteq Nf_1 \bigoplus \cdots \bigoplus Nf_n$ and so is contained in the image of Ψ. As N is a sum of indecomposables this shows that Ψ is epic. Thus, Ψ is an isomorphism. □

One class of modules which satisfy the hypotheses of Proposition 1 are those whose endomorphism rings are row finite matrices over the endomorphism rings of the indecomposable summands. Modules I with the property that the endomorphism rings of all their direct sums are row finite matrices over $End(I)$ were studied in [1].

Proposition 2. *Let M be an injective module which is a direct sum of indecomposable submodules. If N is a module generated by indecomposable summands and there is an IP-isomorphism $\Phi : End(M) \to End(N)$ then $M \cong N$.*

Proof. Let $M = \bigoplus I_\delta$ be a decomposition into indecomposables. By the proof of Proposition 1, N contains an isomorphic copy $M' = \bigoplus Nf_\delta$ of M. If $M' \neq N$ then since M' is injective, it is a summand of N, so $N = M' \bigoplus N'$ for some nonzero N'. But N' gives rise to an idempotent $f \in End(N)$ which is orthogonal to all the f_δ's. This implies that there is a idempotent in $End(M)$ which is orthogonal to all the e_δ – which is a contradiction. □

Let S be a summand of a module M, $\Phi : End(M) \to End(N)$ an isomorphism and T the summand of N produced by S and Φ. We say that Φ has the *B–K property with respect to S* if there is an isomorphism $\Psi : S \to T$ such that for every $a \in End(S)$, $(a)\Phi = \Psi^{-1} a \Psi$.

Proposition 3. *Let $M = \bigoplus I_{ij}$ be a direct sum of indecomposables with the finite embedding property such that $I_{ij} \cong I_{kl}$ if, and only if, $i = k$. Let N be generated by indecomposable summands. If $\Phi : End(M) \to End(N)$ is an IP-isomorphism which, for each i, has the B–K property with respect to a summand I_{ij}, then there is an isomorphism between M and N which induces Φ.*

Proof. For ease of notation, we consider only the case when one isomorphism class of indecomposable appears in the decomposition of M, say $M = \bigoplus I_\alpha$. The general case is a consequence. Then by Proposition 1, M and N are isomorphic so there is a decomposition $N = \bigoplus J_\alpha$ where all I_α's and J_α's are isomorphic. Let I_0 be one of the I_α's and J_0 the corresponding J_β. Let Ψ_0 be an isomorphism $I_0 \to J_0$ which induces Φ on I_0. Let $z_{0\alpha}$ be an isomorphism $I_0 \to I_\alpha$ regarded as an endomorphism of M. Let $u_{\alpha 0} = z_{0\alpha}^{-1}$. Then $\Psi_\alpha = u_{\alpha 0} \Psi_0 (z_{0\alpha} \Phi)$ is an isomorphism $I_\alpha \to J_\alpha$. These then induce an isomorphism $\Psi : M \to N$. Clearly, Ψ is monic. It is epic since, by the proof of Proposition 1, $N = \bigoplus N z_{0\alpha} \Phi$.

We now show that Ψ induces Φ. Let $u_{\alpha\beta}: I_\alpha \to I_\beta$ be any homomorphism. Then $u_{\alpha\beta} = z_{0\alpha}^{-1} v_0 z_{0\beta}$ for some endomorphism v_0 of I_0. Now

$$\begin{aligned}\Psi_\alpha^{-1} u_{\alpha\beta} \Psi_\alpha &= \Psi_\alpha^{-1} z_{0\alpha}^{-1} (z_{0\beta} \Psi_\beta) = \Psi_\alpha^{-1} z_{0\alpha}^{-1} v_0 z_{0\beta} z_{0\beta}^{-1} \Psi_0 (z_{0\beta}\Phi) \\ &= (z_{0\alpha}\Phi)^{-1} \Psi_0^{-1} v_0 \Psi_0 (z_{0\beta}\Phi) = (z_{0\alpha}\Phi)^{-1} (v_0\Phi)(z_{0\beta}\Phi) = u_{\alpha\beta}\Phi\end{aligned}$$

as required. □

Theorem 4. *Let $M = M_0 \bigoplus M_1$ where M_0 and M_1 are direct sums of indecomposables and M_1 is injective. Assume that for one decomposition into indecomposables, M_0 has the finite embedding property. If, for a module N which is generated by indecomposable summands, there is an IP-isomorphism $\Phi: End(M) \to End(N)$ then M and N are isomorphic. The isomorphism $M \to N$ induces Φ if, and only if, from each isomorphism class of indecomposable summands of M there is an indecomposable summand which has the B–K property with respect to Φ.*

Proof. The result follows immediately from Propositions 1–3 and the fact that Φ preserves homogeneous components. □

As a corollary we get the Baer–Kaplansky theorem for torsion abelian groups which are direct sums of indecomposable subgroups.

Corollary 5. *If A and B are torsion abelian groups which are direct sums of indecomposables and $\Phi: End(A) \to End(B)$ is an isomorphism, then there is an isomorphism $\Psi: A \to B$ such that $(\alpha)\Phi = \Psi^{-1}\alpha\Psi$ for each $\alpha \in End(A)$.*

Proof. An indecomposable torsion abelian group is either a cyclic group of order p^n, for a prime p, or the quasi-cyclic group $\mathbb{Z}(p^\infty)$, which is injective. Hence, the reduced part of the group has the finite embedding property. The endomorphism ring of the former is $\mathbb{Z}(p^n)$ and of the latter it is the ring of p-adic integers. Both of these rings only admit the identity map as an automorphism. There are no homomorphisms between p-groups and q-groups, if $p \neq q$. It follows that Φ is an IP-isomorphism and that each indecomposable summand of A has the B–K property with respect to Φ. □

Applying Theorem 4 is simplified by the next result.

Proposition 6. *Let M satisfy the hypotheses of Theorem 4 and let G denote $End(M)$. Then $\Phi: G \to G$ is an IP-isomorphism if, and only if, for every primitive idempotent $e \in G$, $Ge \cong_G G(e\Phi)$. Moreover, Φ has the B–K property with respect to the summand Me of M if, and only if, every ring automorphism of eGe is inner.*

Proof. Let $e, f \in G$ be primitive idempotents. We want to show that $Me \cong Mf$ if, and only if, $Ge \cong_G Gf$. The former implies that there are endomorphisms a, b of M such that $ea = f$ and $fb = e$. This means that $Gea = Gf$ and so $Ge \cong_G Gf$, since Ge

is indecomposable and Gf is projective. Conversely, there are $a \in eGf$ and $b \in fGe$ such that $ea = f$ and $fb = e$. It follows that $Me \cong Mf$.

Let $e \in G$ be a primitive idempotent. First, let us assume that Φ is induced by a linear isomorphism σ from Me. If α is a linear endomorphism of Me, then it is right multiplication by an element a of eGe. Its image $\alpha\Phi$ is an endomorphism of Mf, for the primitive idempotent $f = e\Phi$. Now $\alpha\Phi$ is right multiplication by an element a' of fGf and σ is right multiplication by an element $c \in eGf$. By assumption $\alpha\Phi = \sigma^{-1}\alpha\sigma$ so $a' = c^{-1}ac$ which means that Φ is an inner isomorphism of $eGe \cong fGf$.

Now assume that every automorphism of eGe is an inner and let $a \in eGe$. If $f = e\Phi$ then Φ is an isomorphism between eGe and fGf. Therefore, $a\Phi = c^{-1}ac$ for a $c \in eGf$. But c is just a homomorphism $Me \to Mf$, as required. □

A ring has *finite identity* (element) if its identity is a sum of orthogonal primitive idempotents. A ring R with finite identity is *triangular* if there is a decomposition $1 = e_1 + \cdots + e_n$ of its identity into orthogonal primitive idempotents such that $e_iRe_j = 0$ whenever $i < j$. A module is *primitive cyclic* if it is a homomorphic image of an ideal generated by a primitive idempotent. The next result suggests that there is little hope for the classical Baer–Kaplansky Theorem to apply to large classes of rings and modules.

Proposition 7. *Let R be a basic ring which is Noetherian or triangular. If any two primitive cyclic R-modules are isomorphic whenever their endomorphism rings are isomorphic, then R is a product of nonisomorphic rings with no (proper) idempotents.*

Proof. Let $1 = e_1 + \cdots + e_n$ be a decomposition of the identity of R into orthogonal primitive idempotents. Assume that $e_iRe_j \neq 0$ for some i,j, with $i \neq j$.

If R is triangular then $e_jRe_i = 0$ and so Re_j and Re_j/Re_iRe_j have isomorphic endomorphism rings (namely, e_jRe_j), but they are clearly not isomorphic. Therefore, $e_iRe_j = 0$ for all i,j. That is R is a product of the rings e_iRe_i each of which has no idempotents.

Now assume that R is Noetherian and that $e_jRe_i \neq 0$. Then Re_i has a homomorphic image of Re_j. Let L be such an image, under a map π, and assume that it is not contained in any other image of Re_j. Since Re_i is Noetherian, such an image exists. Let C be a complement of L. Then Re_i/C contains a copy L. Let K be a maximal subideal of Re_j which is not killed by π. Then $Re_i/(L + C)$ and $Re_i/(K\pi + C)$ have isomorphic endomorphism rings. They are not isomorphic because the latter has socle isomorphic to Re_j/K and the former does not. □

Proposition 6 suggests that we may regard the Baer–Kaplansky Theorem as being a statement about the automorphisms of rings. Namely, that the automorphisms of a ring are inner IP-isomorphisms if the ring is the ring of endomorphisms of a torsion abelian group. The problem of classifying all rings whose automorphisms are inner IP-isomorphisms then arises. Such rings are of interest in their own right since their ring and linear (module) structures are closely connected. One would expect these rings to have strong properties. In view of Proposition 7 this may be a more interesting problem then the one we originally addressed.

It is well known that a module may have two direct sum decompositions into indecomposables such that none of the indecomposables in one are isomorphic to any in the other. Consequently, modules which decompose uniquely into indecomposables are of particular interest. However, it is not known which modules have this property, but if the indecomposable summands have local endomorphism rings then their direct sums satisfy the condition (the Krull–Schmidt–Azumaya Theorem [9, Theorem 2.9.17]). If the ring is commutative or (left) Noetherian and the indecomposables are Artinian then the condition holds [10]. But it is not sufficient that the indecomposable summands be uniserial [3] or even Artinian [4].

We now look at some rings whose modules decompose uniquely into indecomposables. We study the question raised above and use it to construct new classes of modules satisfying the Baer–Kaplansky Theorem.

Theorem 8. *Let D be a sfield, N a positive integer greater than 1, R the ring of upper triangular $N \times N$-matrices over D and let J be the radical of R. If I is a (possibly trivial) two-sided ideal of R contained in J^2, then R/I has the property that all its (ring) automorphisms are IP-isomorphisms. In fact, every (left) ideal of R/I is isomorphic to its image under every (ring) automorphism. Every automorphism of R/I is induced by a linear automorphism if, and only if, the only automorphisms of D are the inner ones.*

Proof. To show the simplicity of the idea behind the proof we treat the case when $I=0$ separately. R is an Artinian serial ring with N isomorphism classes of indecomposable projectives whose composition lengths vary from 1 to N. Since an automorphism of R must preserve composition length, it must map an indecomposable projective onto one which is isomorphic to it. Hence, it is an IP-isomorphism.

We now consider the general case. Denote the matrix whose only nonzero entry is 1 at the place (i,i) by e_i. Let $I \subseteq J^2$ be a two-sided ideal of R and let $a=\Sigma_{ij}a_{ij}$ be an element of I, where $a_{ij} \in e_iRe_j$. So both $Ra_{ij}=Ra_{ij}e_j$ and $a_{ij}R=e_ia_{ij}R$ are in I. The former is the column ideal starting at (i,j) and the latter is the row ideal starting at (i,j). Hence, every column ideal starting at (i,k) for $k \geq j$, is also in I. Therefore, I is a set of all blocked matrices with the above properties. Since $I \subseteq J^2$, both Re_1 and Je_i, for $i \geq 2$, are not in I.

Let * denote the coset of an element or set with respect to I and let Φ be an automorphism of R^*. Then $R^*e_1^*$ is the only simple projective (up to isomorphism) in R^* so it is isomorphic to its image under Φ. $R^*e_2^*$ is the summand whose only subideal is isomorphic to $R^*e_1^*$, so $R^*e_2^*$ is isomorphic to its image under Φ. The maximal subideal of $R^*e_{i+1}^*$ is generated by $e_i^*R^*e_{i+1}^*$ and every $e_j^*R^*e_{i+1}^*=e_j^*R^*e_i^*.e_i^*R^*e_{i+1}^*$. These properties are preserved by Φ, so Φ preserves maximal subideals of indecomposable summands. Therefore, Φ preserves the isomorphism type of $R^*e_2^*$, $R^*e_3^*$, and so on, as required.

Let R^*a^* be the image of R^*e^* under the R^*-homomorphism α and let R^*b^* be its image under Φ. Then R^*b^* is an R^*-homomorphic image of $R^*f^* = R^*(e^*\Phi)$ which is R^*-isomorphic to R^*e^*. Since α is right multiplication by a, Φ maps its kernel onto the kernel of $R^*f^* \to R^*b^*$. So if K is the kernel of $R^*e^* \to R^*a^*$ then $L = K\Phi$ is the kernel of $R^*f^* \to R^*b^*$. Since Φ preserves composition length, L and K have the the same length and therefore $R^*a^* \cong_R R^*b^*$. Since R^* is a serial ring all left ideals are direct sums of primitive cyclics, hence they are isomorphic to their images under Φ. The rest of the theorem follows from Proposition 6. □

It is well known [5] that the rings R of Theorem 8 are the basic subrings of the nonsingular Artinian serial rings. Since the properties studied are about indecomposable summands, it follows that Theorem 8 applies to arbitrary nonsingular Artinian serial rings.

However, not all serial rings are so accommodating, as the following example shows.

Example. Let F be a field and V a bivector space over F, one dimensional on both sides. Make V into an algebra by defining the product of any two elements to be zero. Let R be the ring of matrices $\begin{pmatrix} F & V \\ V & F \end{pmatrix}$. Then R is a serial ring with two nonisomorphic uniserial projectives, Re_1 and Re_2, each of length two. The automorphism which interchanges diagonally opposite entries in each matrix is not an IP-isomorphism since it maps Re_1 onto Re_2.

Despite the strong properties enjoyed by the ring R of Theorem 8, we cannot say that that two modules are isomorphic if their endomorphism rings are isomorphic to R. For if T is the direct sum of the quotients of Re_N then $End(T) \cong R \cong End(R)$ but R and T are not isomorphic. Note that the isomorphism $R \to End(T)$ takes monomorphisms to epimorphisms with nonzero kernels. Nevertheless, we can find a class of R-modules which satisfies the Baer–Kaplansky Theorem.

Assume M decomposes uniquely into indecomposables and let $M = \bigoplus_{i,j} I_{ij}$ be such a decomposition, where $I_{ij} \cong I_{st}$ if, and only if, $i = s$. Let $M_i = \bigoplus_j I_{ij}$. We call the M_i the *homogeneous components* of M. We call a module *basic* if all its homogeneous components are indecomposable. Note that a basic submodule of the regular module (the ring) is not a basic subring, although the two play similar roles for many purposes. As we will study Artinian serial rings, we need only consider basic modules (which are necessarily finitely generated) over basic subrings.

Theorem 9. *For the ring R of Theorem* 8, *let $\mathscr{F}$ be the category of R-modules which have a summand isomorphic to R. Then modules in $\mathscr{F}$ are determined, up to isomorphism, by their endomorphism rings. Moreover, every isomorphism between the endomorphism rings of two modules in $\mathscr{F}$ is induced by an isomorphism between the modules if, and only if, the only automorphisms of D are inner.*

Proof. Let M and N be two modules in $\mathscr{F}$ and $\Phi : G = End(M) \to H = End(N)$ an isomorphism. We know that M is a direct sum of uniserials and that this decomposition is unique – up to isomorphism. Assume M is basic. Since it has a copy of R as a summand, M is a direct sum of R and uniserials U_i whose elements are singular. So there are no homomorphisms from the U_i to R and therefore R is a left ideal of both G and H. Since each U_i is an image of an Re the only simple projective in G is the one in R. The argument in the proof of Theorem 8 then shows that Φ is an IP-isomorphism with respect to these copies of R in G and H. A uniserial U is determined by its projective cover Re_i, say, and the kernel K of the defining surjection π. Now K is a copy of Re_k, where $k \leq i$, so π kills all homomorphisms $Re_k \rightarrow Re_i$ but not those from Re_{k+1} to Re_i. That is, U is determined by homomorphisms from the Re_j's to M. By Theorem 8, Φ preserves the isomophism types of left ideals of R. It also preserves the above homomorphisms which define U. Hence, $N(1_U\Phi) \cong_R U$. That is, Φ is an IP-isomorphism. The rest follows from Theorem 4 and Proposition 6. □

A bounded abelian p-group contains a summand $\mathbb{Z}(p^n)$ of maximal length. It is therefore a module over $\mathbb{Z}(p^n)$. So Theorem 9 is Baer's original theorem for R-modules. Since there are no "unbounded" R-modules, it can be regarded as an extension of the Baer–Kaplansky Theorem to nonsingular Artinian serial rings.

Corollary 10. *Let A be the endomorphism ring of a module in the category $\mathcal{F}$ of Theorem 9. Then every automorphism of A is an IP-isomorphism. It is induced by a linear automorphism of A if, and only if, all the automorphisms of D are inner.*

Proof. Apply Proposition 6 and Theorem 9. □

The structure of the ring A (of Corollary 10) is considerably more complicated than that of the upper triangular matrix ring R. A description of A's ideal lattice is given in [6].

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